Math 10B with Professor Stankova
Worksheet, Discussion #23; Tuesday, 4/17/2018
GSI name: Roy Zhao

1 Integrating Irreducible Quadratics

1.1 Concepts

1. For partial fractions, we often end up with something of the form $\frac{Ax+B}{x^2+bx+c}$ where the bottom is an irreducible polynomial, one that does not have real roots, and we need to integrate it. We will always end up with a $\ln x^2+bx+c$ term and an arctan term. In order to split this, let $u = x^2+bx+c$ so $du = 2x+b$ and write $Ax+B = Cdu+D$. Then you need to be able to complete the square and use the fact that

$$\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\arctan(x/a)+C.$$

1.2 Example

2. Find $\int \frac{2x+3}{4x^2+4x+5}dx$.

Solution: First we want to get the denominator to be monic or have a leading term x^2. This gives $\int \frac{2x+3}{4x^2+4x+5}dx = \frac{1}{4}\int \frac{2x+3}{x^2+x+5/4}dx$. Now let $u = x^2+x+5/4$. Then $du = 2x+1$ and we want to write $2x+3 = C(2x+1)+D$. Matching coefficients gives $C = 1$ and $D = 2$. Thus

$$\int \frac{2x+3}{4x^2+4x+5} = \frac{1}{4}\left[\int \frac{du}{u} + \int \frac{2}{x^2+x+5/4}dx\right]$$

The first integral is just $\ln|u| = \ln|x^2+x+5/4|$. For the second integral, we need to complete the square. To do this, take the middle term 1, divide by 2, then square to get 1/4 and so $x^2+x+5/4 = x^2+x+1/4+1 = (x+1/2)^2+1$. Thus

$$\frac{1}{4}\left[\int \frac{du}{u} + \int \frac{2}{x^2+x+5/4}dx\right] = \frac{1}{4}\left[\ln|x^2+x+5/4| + \int \frac{2}{(x+1/2)^2+1}dx\right]$$

$$= \frac{\ln|x^2+x+5/4|}{4} + \frac{\arctan(x+1/2)}{2} + C.$$

1.3 Problems

3. Find $\int \frac{4x+1}{x^2+4x+8}dx$.

Solution: Let $u = x^2+4x+8$ so $du = 2x+4$ and $4x+1 = 2(2x+4)-7$. Then we want to complete the square and we do this by halving 4 to get 2 then squaring to get 4 to get $x^2+4x+8 = x^2+4x+4+4 = (x+2)^2+4$ and hence

$$\int \frac{4x+1}{x^2+4x+8}dx = \int \frac{2(2x+4)}{x^2+4x+8}dx - \int \frac{7}{(x+2)^2+2^2}dx$$

$$= 2\ln|x^2+4x+8| - \frac{7}{2}\arctan((x+2)/2) + C.$$

4. Find $\int \frac{3x+4}{x^2-2x+10}dx$.

Solution: Let $u = x^2-2x+10$ so then $du = 2x-2$ and then $3x+4 = \frac{3}{2}(2x-2)+7$ and to complete the square, we add $(-2/2)^2 = 1$ to get $x^2-2x+10 = x^2-2x+1+9 = (x-1)^2+3^2$. Thus we split to get

$$\int \frac{3x+4}{x^2-2x+10}dx = \frac{3}{2}\int \frac{(2x-2)dx}{x^2-2x+10} + 7\int \frac{1}{(x-1)^2+3^2}dx$$

$$= \frac{3}{2}\ln|x^2-2x+10| + \frac{7}{3}\arctan((x-1)/3) + C.$$

1.4 Extra Problems

5. Find $\int \frac{x-3}{x^2-4x+5}dx$.

Solution: We write

$$\int \frac{x-3}{x^2-4x+5}dx = \frac{1}{2}\int \frac{2x-4}{x^2-4x+5} - \int \frac{1}{(x-2)^2+1}dx$$

$$= \frac{1}{2}\ln|x^2-4x+5| - \arctan(x-2) + C.$$

6. Find $\int \frac{5-x}{x^2-x+1}dx$.

Solution: We write

$$\int \frac{-x+5}{x^2-x+1} = \frac{-1}{2}\int \frac{2x-1}{x^2-x+1}dx + \frac{9}{2}\int \frac{1}{(x-1/2)^2+3/4}$$

$$= \frac{-1}{2}\ln|x^2-x+1| + \frac{9}{2\sqrt{3}/2}\arctan((x-1/2)/(\sqrt{3}/2)) + C$$

$$= \frac{-1}{2}\ln|x^2-x+1| + 3\sqrt{3}\arctan\left(\frac{2x-1}{\sqrt{3}}\right) + C.$$

2 Determinants and Inverses

2.1 Concepts

7. There are 2 ways to calculate the determinant of a 3×3 matrix $\begin{pmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \\ b_7 & b_8 & b_9 \end{pmatrix}$. The first is expansion along the first row to get the determinant is $b_1\begin{vmatrix} b_5 & b_6 \\ b_8 & b_9 \end{vmatrix} - b_2\begin{vmatrix} b_4 & b_6 \\ b_7 & b_9 \end{vmatrix} + b_3\begin{vmatrix} b_4 & b_5 \\ b_7 & b_8 \end{vmatrix}$. The other is to use the diagonal method to get $b_1b_5b_9 + b_2b_6b_7 + b_3b_4b_8 - b_1b_6b_8 - b_2b_4b_9 - b_3b_5b_7$.

   We can determine the number of solutions to an equation $A\vec{x} = \vec{b}$ by the determinant of A and that is given below.

$\det(A)$	$\neq 0$	$= 0$
Number of Solutions	1	0 or ∞

2.2 Examples

8. Find the determinant of $\begin{vmatrix} 3 & 2 & 3 \\ -1 & 5 & -3 \\ 7 & -1 & -1 \end{vmatrix}$.

Solution: One option gives us $3(5\cdot(-1)-(-3)\cdot(-1)) - 2((-1)\cdot(-1)-(-3)\cdot 7) + 3((-1)\cdot(-1)-5\cdot 7) = -170$ and the other way gives us $3\cdot5\cdot(-1) + 2\cdot(-3)\cdot7 + 3\cdot(-1)\cdot(-1) - 3\cdot(-3)\cdot(-1) - 2\cdot(-1)\cdot(-1) - 3\cdot5\cdot7 = -170$.

2.3 Problems

9. True **FALSE** It is possible for $A\vec{x} = \vec{0}$ to have no solutions.

> **Solution:** $\vec{x} = \vec{0}$ is a solution so it cannot have no solutions.

10. **TRUE** False If we know that $A\vec{x} = \vec{b}$ has no solutions, then we know what $\det(A)$ is.

> **Solution:** If it has no solutions, then $\det(A) = 0$.

11. Let $A = \begin{pmatrix} 2 & 4 & -1 \\ 2 & 2 & 5 \\ 6 & 8 & 9 \end{pmatrix}$. What is $\det(A)$? How many solutions that $A\vec{x} = \vec{0}$ have?

> **Solution:** The determinant is $2\cdot 2\cdot 9 + 4\cdot 5\cdot 6 + (-1)\cdot 2\cdot 8 - 2\cdot 5\cdot 8 - 4\cdot 2\cdot 9 - (-1)\cdot 2\cdot 6 = 0$. Therefore $A\vec{x} = \vec{0}$ has 0 or ∞ many solutions. But $\vec{x} = \vec{0}$ is a solution so it cannot have 0 solutions, so it has ∞ many solutions.

12. Let $A = \begin{pmatrix} 0 & 1 & 0 \\ 5 & 1 & 3 \\ 2 & 2 & 4 \end{pmatrix}$. What is $\det(A)$? How many solutions does $A\vec{x} = \vec{0}$ have?

> **Solution:** The determinant is $-\begin{vmatrix} 5 & 3 \\ 2 & 4 \end{vmatrix} = -(5\cdot 4 - 3\cdot 2) = -14$. Since $\det(A) \neq 0$, this has a unique solution and it is $\vec{x} = \vec{0}$.

13. Find the solution to $x + 2y = 3$ and $4x + 5y = 6$ using matrix vector form.

> **Solution:** We write it as
> $$\begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}.$$
> Thus
> $$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}^{-1} \begin{pmatrix} 3 \\ 6 \end{pmatrix} = \frac{1}{1\cdot 5 - 2\cdot 4} \begin{pmatrix} 5 & -2 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 6 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

14. Write $x + y = 10, y + z = 5, x + z = -1$ in matrix vector form. How many solutions does it have?

Solution:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10 \\ 5 \\ -1 \end{pmatrix}.$$

To determine the number of solutions, we can calculate the determinant and it is $1 \cdot 1 \cdot 1 + 1 \cdot 1 \cdot 1 - 0 = 2 \neq 0$. Therefore, there is a unique solution.